The boundary of the complex of free factors

Mladen Bestvina and Patrick Reynolds

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Abstract

We give a description of the boundary of a complex of free factors that is analogous to E. Klarreich's description of the boundary of a curve complex.

1 Introduction

The complex of free factors, denoted $\mathcal{F} = \mathcal{F}_N$, for the free group F_N is an analogue for the complex of curves for a surface. The simplicial complex $\mathcal{F}$ arises as the nerve of the intersection pattern for thin regions in Outer space, and hence codes the geometry of Outer spaces relative to these thin regions. Vertices of $\mathcal{F}$ are conjugacy classes of non-trivial proper free factors of the rank-N free group F_N, and higher dimensional simplices correspond to chains of inclusions of free factors.

Equip $\mathcal{F}$ with the simplicial metric. It was shown by Bestvina and Feighn in [3] that $\mathcal{F}$ is Gromov hyperbolic; the point of the present note is to give a concrete description of the boundary $\partial\mathcal{F}$ of $\mathcal{F}$. Kapovich-Rafi [19] have shown that hyperbolicity of $\mathcal{F}$ can be deduced from the hyperbolicity of the free splitting complex, which was shown by Handel-Mosher [13], and an alternative proof of this was given by Hilion-Horbez [14].

Let $\partial \mathrm{cv}_N$ denote the boundary of the Culler-Vogtmann Outer space; the points of $\partial \mathrm{cv}_N$ are represented by very small actions of F_N on $\mathbb{R}$-trees. Associated to $T \in \partial \mathrm{cv}_N$ is a (algebraic) lamination $L(T)$, which intuitively records information about which elements of F_N act with short translation length in T. A lamination is an F_N-invariant, flip invariant, closed subset $X \subseteq \partial^2 F_N = \partial F_N \times \partial F_N \smallsetminus diag$. A finitely generated subgroup $H \leq F_N$ is a virtual retract of F_N by M. Hall's Theorem, hence H is quasi-convex in F_N, and ∂H embeds in ∂F_N; say that H carries a leaf of X if $X \cap \partial^2 H \neq \emptyset$.

A lamination X is called arational if no leaf of X is carried by a proper free factor of F_N; a tree $T \in \partial \mathrm{cv}_N$ is called arational if $L(T)$ is arational. Let

$\mathcal{AT} \subseteq \partial \mathrm{cv}_N$ denote the set of arational trees, equipped with the subspace topology. Define a relation $\sim$ on $\mathcal{AT}$ by $S \sim T$ if and only if $L(S) = L(T)$, and give $\mathcal{AT}/\sim$ the quotient topology. Our main result is:

Theorem 1.1. *The space $\partial\mathcal{F}$ is homeomorphic to $\mathcal{AT}/\sim$.*

This theorem is a very strong analogue of E. Klarreich's description of the boundary $\partial\mathcal{C}(S)$ of the complex of curves $\mathcal{C}(S)$ associated to a non-exceptional surface S; Klarreich showed that $\partial\mathcal{C}(S)$ is homeomorphic to $\mathcal{AF}/\sim$, where $\mathcal{AF} \subseteq \mathcal{PML}(S)$ is the subspace consisting of arational measured foliations, and where for $E, F \in \mathcal{AF}$ one has $E \sim F$ if and only if the underlying topological foliations are equivalent [20].

The argument for our main result follows the outline of Klarreich's paper, but the details are quite different; the difficulty comes from pushing the analogy between Outer space and Teichmüller space.

The paper is organized as follows. Relevant background about Outer space, very small F_N-trees, laminations, and $\mathcal{F}$ is found in Section 2. The proof of the main result can be roughly divided into four steps. The first step is to show that arational trees are indeed very close analogues of arational measured foliations on surfaces; this is accomplished in Sections 3 and 4. The main result is Theorem 4.4, a duality result, which gives an extremely strong analogy between arational trees and arational measured foliations on surfaces. The second step is to obtain control over the way that trees can fail to be arational; this is accomplished in Sections 5 and 6. Here, we bring a study of standard geodesics in Outer space, which serve as surrogates for Teichmüller geodesics, and the main result there is Lemma 6.13, which shows that if G_t is a folding line converges to a tree $T \notin \mathcal{AT}$, then the image of G_t for large t in $\mathcal{F}$ is a uniformly bounded set. The last two steps involve running Klarreich's argument and collecting some basic facts about the point set topology of the spaces $\mathcal{AT}$, $\mathcal{AT}/\sim$, and $\partial\mathcal{F}$. This is the content of Section 7, where the main result is proved.

Note: Very recently, the main result of this paper was also announced by Hamenstädt [12].

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2 Background

Let F_N denote the free group of rank N. Throughout, we consider isometric actions of F_N on $\mathbb{R}$-trees; all actions are assumed minimal. Let T be a tree; a subset $I \subseteq T$ is called an *arc* if I is isometric to a segment in $\mathbb{R}$. An arc is *non-degenerate* if it contains more than one point. For a subset Y of an F_N-tree T, the stabilizer of Y, denoted $Stab(Y)$, is the set wise stabilizer of Y. In this section, we collect some definitions and basic results.

2.1 Very Small F_N-trees

A subset $Y \subseteq T$ that is the convex hull of three points is called a *tripod* if Y is not a segment. An action $F_N \curvearrowright T$ on a tree T is *very small* if for any non-degenerate arc $I \subseteq T$, either $\mathrm{Stab}(I) = \{1\}$ or $\mathrm{Stab}(I)$ is a maximal cyclic subgroup of F_N, and if for any tripod $Y \subseteq T$, $Stab(Y) = \{1\}$. An action $F_N \curvearrowright T$ is *discrete* (or *simplicial*) if the F_N-orbit of any point of T is a discrete subset of T.

The *unprojectivised Outer Space* of rank N, denoted cv_N, is the topological space whose underlying set consists of free, minimal, discrete, isometric actions of F_N on $\mathbb{R}$-trees. A minimal F_N-tree is completely determined by its translation length function [5]; this gives an inclusion $\mathrm{cv}_N \subseteq \mathbb{R}^{F_N}$ and a topology on cv_N. The non-trivial points in the closure $\overline{\mathrm{cv}}_N$ in $\mathbb{R}^{F_N}$ are very small isometric actions of F_N on $\mathbb{R}$-trees [6, 1]. The Culler-Vogtmann Outer space, denoted CV_N, is the image of cv_N in the projective space $\mathbb{PR}^{F_N}$; the points of CV_N are thought of as free, simplicial F_N-trees of co-volume one. The group $Out(F_N)$ acts on cv_N, $\overline{\mathrm{cv}}_N$, and CV_N: given a tree T with length function l_T and an element $\Phi \in Out(F_N)$, for $g \in F_N$, set $l_{T\Phi}(g) := l_T(\varphi(g))$, where φ is any lift of Φ to $Aut(F_N)$.

Let $\partial\mathrm{cv}_N := \overline{\mathrm{cv}}_N \setminus \mathrm{cv}_N$, so points of $\partial\mathrm{cv}_N$ are very small F_N-trees that are either non-free or non-simplicial. Let $T \in \partial\mathrm{cv}_N$, and let $H \leq F_N$ be finitely generated. If H does not fix a point in T, then we let T_H stand for the minimal H-invariant subtree of T; T_H is the union of axes of hyperbolic elements of H. If T has trivial arc stabilizers, which is always the case when T has dense orbits, then for any finitely generated $H \leq F_N$, there is a unique minimal tree for H: either T_H in the case of the previous sentence, or the unique fixed point of H, if H contains no hyperbolic element.

2.2 Algebraic Laminations and Currents

We review algebraic laminations associated to F_N-trees; see [9] and [10] for details. Let ∂F_N denote the Gromov boundary of F_N — *i.e.* the Gromov boundary of any Cayley graph of F_N; Gromov boundaries are reviewed below. Let $\partial^2(F_N) := \partial F_N \times \partial F_N \smallsetminus \Delta$, where Δ is the diagonal. The left action of F_N on a Cayley graph induces actions by homeomorphisms of F_N on ∂F_N and $\partial^2 F_N$. Let $i : \partial^2 F_N \to \partial^2 F_N$ denote the involution that exchanges the factors. A *lamination* is a non-empty, closed, F_N-invariant, i-invariant subset $L \subseteq \partial^2 F_N$.

Associated to $T \in \partial \mathrm{cv}_N$ is a lamination $L(T)$, which is constructed as follows. Let $L_\epsilon(T) := \{(g^{-\infty}, g^{\infty}) | l_T(g) < \epsilon\}$, so L_ϵ consists of the pairs of fixed points in ∂F_N of elements $g \in F_N$ with short translation length in T. Define $L(T) := \cap_{\epsilon > 0} L_\epsilon$, then $L(T)$ is a lamination.

Let $T \in \partial \mathrm{cv}_N$, and let $H \leq F_N$ be finitely generated. Then H is virtually a retract of F_N and, hence, is quasi-convex in F_N; so $\partial^2 H$ embeds in $\partial^2 F_N$. We say that H *carries* a leaf of $L(T)$ if there is a leaf $l \in L(T)$ such that $l \in \partial^2 H$. We note that H carries a leaf of $L(T)$ if and only if either some element of H fixes a point in T, or the action $H \curvearrowright T_H$ is not discrete.

A *geodesic current* is an F_N-invariant Radon measure ν on $\partial^2 F_N$, *i.e* ν is Borel measure that is finite on compact subsets of $\partial^2 F_N$. Let $Curr(F_N)$ denote the set of currents; equip $Curr(F_N)$ with the weak* topology. The group $Out(F_N)$ acts on $Curr(F_N)$ on the left: let $C \subseteq \partial^2 F_N$ be compact, let $\Phi \in Out(F_N)$, and let $\nu \in Curr(F_N)$, then $\Phi(\nu)(C) := \nu(\varphi^{-1}(C))$, where $\varphi \in Aut(F_N)$ is any lift of Φ.

If $g \in F_N$ is such that the conjugacy class of g does not contain an element of the form h^k for $h \in F_N$ and $k > 1$, then there is a *counting current*, denoted η_g, associated to the conjugacy class of g. In [17], Kapovich and Lustig establish the following:

Proposition 2.1. *[17, Theorem A] There is a unique $Out(F_N)$-invariant, continuous length pairing that is $\mathbb{R}_{\geq 0}$ homogeneous in the first coordinate and $\mathbb{R}_{\geq 0}$-linear in the second coordinate*

$$\langle \cdot, \cdot \rangle : \overline{cv}_N \times Curr(F_N) \to \mathbb{R}_{\geq 0}$$

Further, $\langle T, \eta_g \rangle = l_T(g)$ for all $T \in cv_N$ and all counting currents η_g.

The support $Supp(\nu)$ of a current ν is a lamination on F_N; $Supp(\nu)$ has an isolated point if and only if ν has an atom. Kapovich and Lustig give the following characterization of zero intersection:

Proposition 2.2. *[18, Theorem 1.1] Let $T \in \overline{cv}_N$, and let $\nu \in Curr(F_N)$. Then $\langle T, \nu \rangle = 0$ if and only if $Supp(\nu) \subseteq L(T)$.*

We let $\mathbb{P}Curr(F_N)$ denote the space of projective classes (*i.e.* homothety classes) of currents. The action of $Out(F_N)$ on $\mathbb{P}Curr(F_N)$ is not minimal, but there is a unique minset $\mathbb{P}M_N \subseteq \mathbb{P}Curr(F_N)$ that is the closure of projective currents corresponding to primitive conjugacy classes of F_N [16]; let M_N denote the preimage of $\mathbb{P}M_N$ in $\mathbb{P}Curr(F_N)$.

2.3 Gromov Hyperbolic Spaces

We give a very brief review of Gromov hyperbolic spaces and their boundaries. Let (X, d) be a metric space, and let $p \in X$ be a basepoint. For $x, y \in X$, the *Gromov product* of x and y (relative to p) is defined as

$$(x, y) = (x, y)_p := \frac{1}{2}(d(x, p) + d(y, p) - d(x, y))$$

The metric space (X, d) is called *Gromov hyperbolic* if there is some $\delta \geq 0$ such that for any $x, y, z \in X$, one has

$$(x, z) \geq \min\{(x, y), (y, z)\} - \delta$$

If (X, d) is a geodesic metric space, then hyperbolicity of (X, d) also can be characterized by geodesic triangles being *thin*.

If (X, d) is Gromov hyperbolic, then one says that a sequence of points $\{x_n\}$ *converges* if $(x_n, x_m) \to \infty$ as $m, n \to \infty$. Two convergent sequences $\{x_n\}, \{y_n\}$ are *equivalent* if $(x_n, y_n) \to \infty$. The *boundary* ∂X of X is defined to be the collection of equivalence classes of convergent sequences in X; two equivalence classes of sequences are close in ∂X if any pair of representatives have large Gromov product for all large n. That all this is well-defined follows from hyperbolicity.

Given metric spaces (X, d) and (X', d') and a number C, a function $f : X \to X'$ is called a *C-quasi-isometric embedding* if for all $x, y \in X$

$$\frac{1}{C}d(x, y) - C \leq d'(f(x), f(y)) \leq Cd(x, y) + C$$

The map f is a *quasi-isometry* if in addition, for any $z' \in X'$, there is $z \in X$ such that

$$d'(f(z), z') \leq C$$

If the spaces X and X' are equipped with an action of a group G, one arrives at the obvious notion of *G-equivariant quasi-isometry*. Any quasi-isometry $X \to X'$ between Gromov hyperbolic spaces induces a homeomorphism $\partial X \to \partial X'$.

A *quasi-geodesic* in X is a quasi-isometrically embedded copy of an interval of $\mathbb{R}$. Two quasi-geodesic rays $r, r' : [0, \infty) \to X$ with $r(0) = r'(0) = 0$ are *equivalent* if their images have finite Hausdorff distance in X. The boundary ∂X coincides with the collection of equivalence classes of quasi-geodesic rays (based at p), where two classes of rays are close if a pair of representatives stay close for a large initial segment of $[0, \infty)$.

2.4 The Complex of Free Factors and Variants

The *complex of free factors*, denoted $\mathcal{F}$, has as vertices conjugacy classes of non-trivial proper free factors of F_N, where conjugacy classes $[A^1], \dots, [A^{k+}]$ span a simplex in $\mathcal{F}$ if and only if there are representatives $A^1, \dots, A^{k_1}$ such that after possibly reordering $A^1 \leq \dots \leq A^{k+1}$. Regard $\mathcal{F}$ as a metric space by identifying each simplex with a standard simplex, and endow the resulting space with the path metric. Being its 1-skeleton, the *graph of free factors* is quasi-isometric to the complex of free factors. We have:

Proposition 2.3. *[3, Main Theorem] The metric space $\mathcal{F}$ is hyperbolic.*

Throughout the sequel, we shall use the term *factor* to mean a conjugacy class of non-trivial, proper free factors of F_N; oftentimes, we will blur the distinction between conjugacy classes and the subgroups representing them, since we expect little confusion to arise from this.

We now introduce some variants of $\mathcal{F}$. First let $\mathcal{F}'$ denote the graph with vertices conjugacy classes of co-rank 1 free factors of F_N, where two vertices are adjacent if they have representatives with non-trivial intersection. It is an easy exercise to check that $\mathcal{F}'$ is ("coarsely F_N-equivariantly") quasi-isometric to $\mathcal{F}$.

Now, let $\mathcal{ST}$ denote the set of Bass-Serre trees for free splittings of F_N with a co-rank 1 vertex stabilizer; when convenient, we regard Bass-Serre trees as metric trees with the simplicial metric. We declare two trees in $\mathcal{ST}$ to be adjacent if there is an element of F_N that fixes a point in both of them. Give $\mathcal{ST}$ the simplicial metric; evidently, $\mathcal{ST}$ is quasi-isometric to $\mathcal{F}'$, so we have:

Lemma 2.4. *The graph $\mathcal{ST}$ is quasi-isometric to $\mathcal{F}$. In particular, $\mathcal{ST}$ is hyperbolic and $\partial \mathcal{ST} \cong \partial \mathcal{F}$.*

There is an equivalent characterization of adjacency in $\mathcal{ST}$: T and T' are adjacent if there is a current η_g corresponding to a primitive conjugacy class g such that $\langle T, \eta_g \rangle = 0 = \langle T', \eta_g \rangle$. The advantage of this equivalent notion of adjacency is that one can pass to limits and use the continuity of the Kapovich-Lustig intersection pairing; see the proof of Corollary 4.5.

There is a coarsely well-defined projection $\pi : \mathrm{cv}_N \to \mathcal{ST}$: for $T \in \mathrm{cv}_N$, choose an edge e of T that projects to a non-separating edge in T/F_N, then collapse every edge of T not in the orbit of e. Then $\pi(T)$ is a set of diameter 1 in $\mathcal{ST}$, provided $N \geq 3$. Abusing notation, we also use the symbol π to denote the following projection $\mathrm{cv}_N \to \mathcal{F}$: associate to $T \in \mathrm{cv}_N$ the collection of free factors represented by subgraphs of T/F_N. It is noted in [3] that $\pi(T)$ has diameter at most 4 and that if the volume of an immersion representing a factor F in T/F_N is uniformly bounded, then $d_{\mathcal{F}}(\pi(T), F)$ is uniformly bounded as well.

Given a number K, say that a function $\iota : [0, \infty) \to X$ is an *unparameterized quasi-geodesic* if there are $0 = t_0 < t_1 < \ldots < t_m < \ldots \in [0, \infty)$ such that $diam(\iota([t_i, t_{i+1}]) \leq K$ and $|i - j| \leq d(\iota(t_i), \iota(t_j)) + 2$. In the statement below, π can be taken to be either of the projections defined above.

Proposition 2.5. *[3, Corollary 5.5] Let G_t be a parameterized standard geodesic in cv_N. Then $\pi(G_t)$ is an unparameterized quasi-geodesic with uniform constant.*

Here and throughout, the phrase *uniform constant* is taken to mean a constant that depend only on $N = Rank(F_N)$. Below, we will define and give a thorough description of *standard geodesics* in cv_N.

2.5 Standard Geodesics

Let $T, U \in \mathrm{cv}_N$. An equivariant function $f : T \to U$ is called *linear on edges* if for any any edge $I \subseteq T$, the restriction of f to I is a linear function onto its image. Any Lipschitz equivariant function $f : T \to U$ is homotopic to a function f' that is linear on edges such that $Lip(f') \leq Lip(f)$. In the sequel, functions $f : T \to U$ will be equivariant and linear on edges, unless forced to be otherwise. An equivariant Lipschitz function $T \to U$ always exists; one chooses a (linear) image in U for a lift of a maximal tree in T/F_N, and equivariance and linearity on edges gives a unique extension $T \to U$. Any two equivariant Lipschitz maps $T \to U$ are homotopic; such a map is usually called a *change of marking.*

Let $f : T \to U$; if f is not a homeomorphism, then f *folds*–there is a point $x \in T$ and an embedded arc $i : [-s, s] \to T : 0 \mapsto x$, such that

$(f \circ i)|_{[-\epsilon,\epsilon]}$ is not an embedding for any $\epsilon > 0$.

A map $f : T \to U$ is called *optimal* (with respect to the pair T, U) if $Lip(f) \leq Lip(g)$ for any $g : T \to U$. Optimal maps exist; the point is that one needs only specify the image of a particular fundamental domain for T, and moving this image far from the (compact set of) fundamental domains for U increases the Lipschitz constant: $\inf Lip(f : T \to U) = \min Lip(f : T \to U)$.

Now suppose that f is optimal, and let $x \in T$; it cannot be the case that every embedding $[-s, s] \to T : 0 \mapsto x$ is folded by f. Indeed, if this were the case, then we can locally modify f by sliding $f(x)$ toward the common endpoint of all (sufficiently long) embeddings $[-s, s] \to T : 0 \mapsto x$. This would decrease $Lip(f)$, contradicting optimality of f.

An immersed path γ in T is called *legal* if f does not fold any portion of γ, *i.e.* if $f \circ \gamma$ is an immersed path in U. A path that is not legal is called *illegal*. If γ is an illegal, there is a point $v \in \gamma$ such that for any subpath $[-s, s] \to \gamma : 0 \mapsto v$, $f([-s, s])$ is not embedded; the pair of germs about 0 of these subpaths is called an *illegal turn*. The previous paragraph gives that all illegal turns occur at vertices of T. A path γ in T is called a *loop* if there is $g \in F_N$ such that $g\gamma(0) = \gamma(1)$. An easy pigeon-hole argument shows that there is a uniform upper bound (depending only on N) for the minimal simplicial length of a legal loop.

By adjusting the metric on T, we can assume that f is isometric when restricted to edges; here is how to do this. Since U is an $\mathbb{R}$-tree, there is an ambient *length measure* on U: segments in U can be identified with segments of $\mathbb{R}$, and hence are equipped with Lebesgue measure. This collection of (local) measures is compatible with restriction and is respected by the (isometric) action $F_N \curvearrowright U$. The measure on U can be pulled-back to T via any equivariant, linear on edges map $f : T \to U$ to get a new tree T_U, and the Lipschitz map f induces a 1-Lipschitz map $f_U : T_U \to U$ that is non-constant on every non-degenerate arc of T_U.

Note that $T_U \in cv_N$ is contained in the same (partial) simplex as T; there is a path in this simplex from T to T_U obtained by taking convex combinations of the two measures on the underlying set of T. It is possible that T_U is not in the same open simplex as T; this occurs exactly when f is constant on some edge. Evidently, the map f_U is isometric on edges and folds isometrically.

The map f gives a path in cv_N from T_U to U: choose a point $x \in T$ and an embedding $[-s, s] \to T : 0 \mapsto x$ such that $f(-s) = f(s)$ and s is maximal (at x) with respect to this feature and such that x is chosen so that s is minimal among s' and $y \in T$ defined as above. Now, we modify T_U by

folding for time s according to f; that is to say, fold all illegal turns for time s. Now, find a new s as above and continue to fold; this either gives a path $T_U \to U$ or there is a limit from which we can continue folding. The result is a path from T_U to U in cv_N that is called a *folding path*. Note that every point $(T_U)_t$ on a folding path $T_U \to U$ comes with an induced equivariant map $f_t : (T_U)_t \to U$. The concatenation of the path $T \to T_U$ with the path $T_U \to U$ is called a *standard geodesic* from T to U in cv_N. More details about all this can be found in [3].

The image of a folding line (resp. standard geodesic) in CV_N is also called a folding line (resp. standard geodesic).

3 Laminations and Dendrites

An F_N-tree $T \in \partial\mathrm{cv}_N$ is called *indecomposable* if for any non-degenerate arcs $I, J \subseteq T$, there are $g_1, \ldots, g_r \in F_N$ such that $I \subseteq g_1 J \cup \ldots \cup g_r J$ and such that $g_i J \cap g_{i+1} J$ is non-degenerate. The point of this section is to prove the following maximality condition about laminations associated to indecomposable trees.

Proposition 3.1. *Let $T \in \partial cv_N$ be indecomposable. If $U \in \partial cv_N$ satisfies $L(T) \subseteq L(U)$, then $L(T) = L(U)$.*

To prove this fact, we will need to consider actions by homeomorphisms of F_N on *dendrites*, which are compact, uniquely arc connected spaces. The connection to actions in $\partial\mathrm{cv}_N$ comes from [8].

The *weak topology*, also called the *observers' topology* in [8], on T has basis the collection of directions at points of T; let T_w denote T with the weak topology. Both the metric and weak topologies extend in obvious ways to give topologies on $\hat{T} = \overline{T} \cup \partial T$, where $\overline{T}$ is the metric completion of T; use $\hat{T}_w$ to denote $\hat{T}$ with the weak topology. The weak topology is weaker than the metric topology, and $\hat{T}_w$ is a dendrite. It is shown in [8] that if T has dense orbits, then the quotient space $\partial F_N / L(T)$ is homeomorphic to $\hat{T}_w$. There is a natural embedding of T_w into $\hat{T}_w$; note that T_w is uniquely arc connected but is not compact. The action of F_N on T induces an action by homeomorphisms on $\hat{T}_w$ for which T_w is invariant.

Note that T_w is the subspace consisting of points of $\hat{T}_w$ that are contained in the interior of an embedded path in $\hat{T}_w$, that is, the set of points x of $\hat{T}_w$ that are separating. Call the points of $\hat{T}_w \smallsetminus T_w$ *endpoints*. Connected subsets of $\hat{T}_w$ are path connected. Since the metric topology agrees with the weak topology on finite subtrees of T, we have that segments in T are

segments in T_w, and tripods in T are tripods in T_w. Hence the action of F_N on the space T_w is very small. Any segment in $\hat{T}_w$ with endpoints in $\hat{T}_w \smallsetminus T_w$ meets T_w in an open dense sub-segment. If T is indecomposable, then so is T_w.

Proposition 3.2. *Let $p : X \to Y$ be a surjective map between two dendrites. Assume that:*

(i) $X = \hat{T}_w$ *for* $T \in \partial cv_N$ *indecomposable,*

(ii) F_N *acts on* Y*, and* p *is* F_N*-equivariant, and*

(iii) Y *has only countably many branch points.*

Then one of the following holds:

(a) p *is a homeomorphism,*

(b) Y *is a point, or*

(c) there is an open interval $Z \subset Y$ *such that for every* $z \in Z$ *we have* $|p^{-1}(z)| > 2$.

Before we begin the proof we will make an observation. Assume that the conclusion of the above proposition fails. Suppose $[a, b]$ and $[c, d]$ are two segments in X with $[a, b] \cap [c, d] = [u, v]$ a nondegenerate segment. If $p(a) = p(b)$ and $p(c) = p(d)$ then $p(u) = p(v)$.

To see this, first note that up to symmetry, there are 3 possible configurations, which are shown in Figure 1.

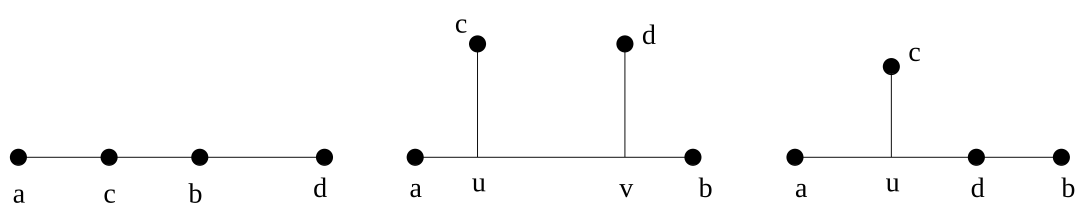

Figure 1:

For the first configuration, if $p(a) = p(b) = r$ and $p(c) = p(d) = s$ but $r \neq s$, then take for Z the open interval (r, s). Every $z \in Z$ has a preimage point in each interval $(a, c), (c, b), (b, d)$.

In the second configuration, if $p(u) \neq p(v)$, take $Z = (p(u), p(v))$ and argue that the preimage of each $z \in Z$ intersects $[a, b]$ in at least two points, and $[c, u] \cup [d, v]$ in at least one point.

In the last configuration, if $p(u) \neq p(d) = p(c)$ take $Z = (p(u), p(d))$. Each $z \in Z$ has at least two preimages in (a, b) and at least one in (c, u).

Proof of Proposition 3.2. First note that if p collapses a nondegenerate segment, then indecomposability of X forces p to be constant, hence Y a point. Hence, assume that p does not collapse any non-degenerate interval and that p is not a homeomorphism. This gives that p is not injective, so there are distinct $a, b \in X$ with $p(a) = p(b)$. After replacing $[a, b]$ by a smaller interval, we may assume that $a, b \in T$ and that $p(a) = p(b)$ has valence 2.

Then there are distinct $c, d \in [a, b]$ with $p(c) = p(d)$ and with the T-distance between c, d arbitrarily small. Apply indecomposability to $I = [a, b]$ and $J = [c, d]$ to deduce that $I \subset \cup g_i(J)$ for $i = 1, 2, \cdots, k$ and $g_i(J) \cap g_{i+1}(J)$ is nondegenerate. We may also assume that k is minimal, so in particular $g_1(J)$ and $g_k(J)$ will contain the endpoints of $[a, b]$. Apply the Observation to the segments I and $g_i(J)$ (by equivariance, the endpoints of $g_i(J)$ are mapped to the same point). Thus the endpoints of $g_i(J) \cap I$ map to the same point y_i in Y. We now claim that $p(a) = p(b) = y_1 = \cdots = y_k$. It is clear that $p(a) = y_1$ since a is an endpoint of $g_1(J) \cap I$ (up to switching a and b) and similarly $p(b) = y_k$. To see that $y_1 = y_2$ apply the Observation to $g_1(J)$ and $g_2(J)$ etc.

We now have points $a = t_0 < t_1 < \cdots < t_m = b$ in $[a, b]$ with $p(t_i) = y$ for every i. We may take m as large as we want by making J small. The images of the intervals $[t_i, t_{i+1}]$ are dendrites D_i containing y, and since the valence of y is 2, as soon as we have $m \geq 3$ two of the dendrites, say D_i and D_j, will have nondegenerate overlap. Take Z to be an open interval in the overlap. Then any point in Z will have at least two preimages in $[t_i, t_{i+1}]$ and at least two in $[t_j, t_{j+1}]$. □

We are now in position to prove Proposition 3.1. For the proof we will need the main result of [7]; we note that if $T \in \partial \mathrm{cv}_N$ has dense orbits, then the map Q used to define *Q-index* in [7] is the quotient map $Q = Q_T : \partial F_N \to \partial F_N / L(T) = \hat{T}_w$.

Proof of Proposition 3.1. Suppose that $U \in \partial \mathrm{cv}_N$ satisfies $L(T) \subseteq L(U)$. It follows from [21] that U can be assumed to have dense orbits; indeed, if U does not have dense orbits, then we can collapse the simplicial part of U to get a tree with dense orbits, and one easily sees from the definition of $L(\cdot)$ that the associated lamination can only be enlarged; see [21] or [24] for details. One has that the quotient map $\partial F_N \to \partial F_N / L(U) = \hat{U}_w$ factors through $\partial F_N \to \partial F_N / L(T) = \hat{T}_w$, so we get a surjective map $p : \hat{T}_w \to \hat{U}_w$, which is F_N-equivariant.

Now apply Proposition 3.2; hypothesis (iii) is satisfied since U (hence $\hat{U}_w$) has countably many branch points [15]. Since U contains more than one point, conclusion (b) is not possible. If conclusion (c) holds, then there are uncountably many points of $\hat{U}$ whose pre-image in ∂F_N contains strictly more than two points; but this is impossible by Theorem 5.3 of [7]. Hence, p is a homeomorphism, so $L(T) = L(U)$. $\square$

4 Arational Trees

We recall a notion of reduction for very small trees, introduced in [24]. For $T \in \partial \mathrm{cv}_N$ and F a factor, say that F *reduces* T if F acts with dense orbits on some subtree $Y \subseteq T$. It should be emphasized that Y can consist of a single point. If Y contains two points, then Y necessarily has infinite diameter, and in this case the minimal subtree T_F for F is dense in Y.

Use $\mathcal{R}(T)$ to denote the set of all factors reducing T. It is noted in [24] that if F' is a factor carrying a leaf of $L(T)$, then there is $F \in \mathcal{R}(T)$ with $F \leq F'$; so, regarded as subsets of $\mathcal{F}$, $\mathcal{R}(T)$ is 1-dense in the set of all factors carrying a leaf of $L(T)$.

When $\mathcal{R}(T) = \emptyset$, we say that T is *arational*; this is equivalent to the statement that no leaf of $L(T)$ is carried by a factor (see [24]). Toward establishing an intuitive analogy with surfaces, we note that the analogous laminations are precisely the arational laminations–*i.e.* the minimal and filling laminations.

Arational trees are a primary object in the present article; we have the following classification:

Proposition 4.1. *[24, Theorem 1.1] Let $T \in \partial cv_N$. The following are equivalent:*

(i) T is arational,

(ii) T is indecomposable, and if T is not free, then T is dual to an arational measured lamination on a surface with one boundary component.

For the convenience of the reader we give a sketch of the proof.

Proof. Step 0: If T is arational, then T has dense orbits, and in particular trivial arc stabilizers.

Step 1: If T is free and indecomposable, then T is arational. Indeed, every f.g. subgroup of F_N with infinite index acts (freely) simplicially on T by [23].

Step 2: If T is arational, then T is mixing. This step is involves some case analysis and uses the Rips theory; see [24] for details.

Step 3: If T is arational and free, then T is indecomposable. By Step 2, T is mixing, and if T is not indecomposable, then T *splits* as a *graph of actions.* There is an associated graph of groups decomposition for F_N which would have trivial edge stabilizers, since T is free. It follows that there is a proper free factor that carries a leaf of $L(T)$, and this implies that there is a (possibly smaller) factor that reduces T, which is not possible.

Step 4: If T is arational, but not free, then T is geometric, and dual to an arational measured lamination on a surface with one boundary component. We give some details here, since the argument is short and accessible. Let $S \to T$ be a geometric resolution where S is not free; note that S has trivial arc stabilizers. Consider a band complex K whose dual tree is S. Then S cannot have minimal components of thin (Levitt) type; indeed, if so one can run Process I of the Rips machine for some time to find a band that disjoint from any loop contained in a leaf, which implies that some point stabilizer in T is contained in a proper free factor. Hence a proper factor carries a leaf of $L(T)$, and we again find a factor reducing T.

Note that S (and K) cannot have simplicial components, since edge stabilizers in S are trivial. It follows that K can be transformed to the following: there are components (vertex stabilizers) V_i disjoint from the lamination, and then surfaces with boundary F_j are attached along boundary components, and each F_j carries a filling lamination. Arguing along the lines of Schenitzer-Swarup (see *e.g.* [1, Lemma 4.1]), one can pull away at least one boundary component of some F_j (i.e. it is attached along points). It now follows that there can be only one surface, which can have only one boundary component, since any other boundary component would be elliptic in S, and hence in T, and would represent a conjugacy class contained in a proper free factor. $\square$

If $X \subseteq \partial^2 F_N$, say that a leaf $l = (x, y) \in \partial^2 F_N$ is *diagonal* over X if there are leaves $(x_1, x_2), (x_2, x_3), \ldots, (x_{r-1}, x_r) \in X$, such that $x = x_1$ and $y = x_r$; and say that X is *diagonally closed* if every leaf that is diagonal over X belongs to X. Laminations associated to trees are always diagonally closed [10].

We collect the following information about laminations associated to arational trees; for the statement, $L'(T)$ denotes the Cantor-Bendixson derivative of $L(T)$, *i.e.* $L'(T) = (L(T))'$ is the set of non-isolated points of $L(T)$, and we set $L''(T) = (L'(T))'$.

Proposition 4.2. *Let $T \in \partial cv_N$.*

(i) *If T is free and indecomposable, then $L'(T)$ is minimal, no leaf of $L'(T)$ is carried by a free factor, and $L(T)$ is obtained from $L'(T)$ by adding finitely many F_N-orbits of isolated leaves, each of which is diagonal and not closed.*

(ii) *If T is dual to an arational measured lamination on a surface with one boundary component, then $L''(T)$ is minimal, no leaf of $L'(T)$ is carried by a free factor, and $L(T)$ is the smallest diagonally closed lamination containing by $L''(T)$. Moreover, $L(T) \smallsetminus L'(T)$ consists of countably many F_N-orbits of diagonal non-periodic leaves and one orbit of a periodic leaf (the conjugacy class of the boundary); these leaves are in the closure of the set of leaves that are diagonal over $L''(T)$.*

Proof. Statement (i) follows from the main results of [23] and [11]. Statement (ii) follows from straightforward considerations about surface laminations and foliations, and the reader is assumed to have some familiarity with this; see [4] and [10] for more details.

Let S be a surface with one boundary component, equipped with a measured foliation (F, μ) to which T is dual. Let I be an arc in S transverse to F. The holonomy along F gives a first return map $I \to I$ that is an interval exchange. Suspend this interval exchange to get a band complex which is homeomorphic to S and with measured foliation equivalent to F; collapsing each band onto one of its leaves gives a homotopy equivalence with a graph. Choose a labeling $a_1, \dots, a_N$ for the oriented edges of this graph; this gives a labeling for all leaves of F, and $L(T)$ is precisely the set of all labels of the lifted foliation $\tilde{F}$ on the universal covering $\tilde{S}$ of S.

The lamination $L'(T)$ consists of an orbit of a periodic leaf l_B, corresponding to the boundary of S, along with leaves that are the labels of leaves of the geodesic lamination L on $\tilde{S}$ got by choosing a hyperbolic structure on S and straightening $\tilde{F}$. Evidently, every leaf of $L(T)$ is diagonal over $L'(T)$.

Since F is arational, the complementary region of $S \smallsetminus L$ containing the boundary of S is a *crown*. There is a leaf l of $L(T)$ that is diagonal over $L''(T)$ that corresponds to entering a small regular neighborhood of the boundary of S from a prong of the crown, following along the boundary, and then exiting back into the same prong. We get leaves, which are diagonal over the set of translates of l, that contain arbitrarily large subwords of l_B. It follows that $L(T)$ is the closure of the diagonal closure of $L''(T)$; statement (ii) follows. □

Hence we get the following:

Corollary 4.3. *Let $T, U \in \partial cv_N$. If T is arational and if $L''(T) \subseteq L(U)$, then $L(T) = L(U)$.*

Proof. Using Proposition 4.2, we get that $L''(T) \subseteq L(U)$ implies that $L(T) \subseteq L(U)$. Apply Proposition 3.1 to conclude. $\square$

The proof of the main result of [11] shows that if T is arational, then the leaves of $L''(T)$ are approximable by primitive conjugacy classes, so for any arational tree T, compactness of $\mathbb{P}M_N$ gives an element $\mu \in M_N$ with $Supp(\mu) \subseteq L''(T)$. The situation directly addressed there is for T free and indecomposable; in this case $L''(T) = L'(T)$. On the other hand, it is clear that the splitting works for non-free arational trees as well. We can now prove a main technical result.

Theorem 4.4. *Let $T \in \partial cv_N$. If T is arational, and if $\mu \in M_N$ satisfies $\langle T, \mu \rangle = 0$, then $Supp(\mu) = L''(T)$. In particular, if U is another very small tree satisfying $\langle U, \mu \rangle = 0$, then U is arational and $L(T) = L(U)$.*

Proof. If $\mu \in M_N$ satisfies $\langle T, \mu \rangle = 0$, then Proposition 2.2 gives that $Supp(\mu) \subseteq L(T)$. The support of a current cannot contain non-periodic isolated leaves, since translates of such leaves have accumulation points.

We now show that the support of a current in M_N cannot contain a periodic leaf corresponding to a conjugacy class not carried by a free factor. Choose a basis for F_N, and let μ_n be a sequence of currents corresponding to primitive conjugacy classes g_n with μ_n converging to μ. Each g_n has Whitehead graph that either is disconnected or has a cut point. After passing to a subsequence, g_n all have the same Whitehead graph W. If l is a periodic leaf of $Supp(\mu)$ corresponding to $g \in F_N$, then for $n >> 0$, g_n contains g^2 as a subword; it follows that the Whitehead graph for g is contained in W, so g is carried by a proper factor. We conclude by Proposition 4.2 that $Supp(\mu) \subseteq L''(T)$, and hence $Supp(\mu) = L''(T)$.

Now let $U \in \partial \mathrm{cv}_N$ be some other tree such that $\langle U, \mu \rangle = 0$. Again by Proposition 2.2, we have that $L''(T) \subseteq L(U)$. Apply Corollary 4.3 to conclude. $\square$

We obtain the following result, which suggests that arational trees lie "at infinity" with respect to $\mathcal{F}$.

Corollary 4.5. *Let $T_n \in cv_N$ be a sequence of trees converging to an arational tree T, and let $Y_n = \pi(T_n)$ denote a projection to $\mathcal{F}$. For any basepoint $0 \in \mathcal{F}$, we have $d(0, Y_n) \to \infty$.*

We follow the outline of Feng Luo.

Proof. According to Lemma 2.4, we can replace $\mathcal{F}$ with $\mathcal{ST}$. Choose a basepoint $0 \in \mathcal{ST}$. Toward contradiction, suppose that $d(0, Y_n)$ does not go to infinity; by passing to a subsequence, we can assume that $d(0, Y_n) = r$ for every n. Then there are trees $0 = T_n^0, T_n^1, \ldots, T_n^{r-1}, T_n^r = Y_n \in \mathcal{ST}$ and currents $\eta_n^1, \ldots, \eta_n^r$ such that $\langle T_n^{i-1}, \eta_n^i \rangle = 0 = \langle T_n^i, \eta_n^i \rangle$. After possibly passing to a further subsequence and rescaling, we can assume that the other T_n^i's and the η_n^i's converge to non-trivial trees T^i and currents η^i, respectively. By Theorem 4.4 each T^i is arational; on the other hand, as the support of each η_n^1 is carried by a fixed free factor of F_N, the same is true for η^1, a contradiction. □

5 Primitive Elements and Vertex Groups

By a *vertex group* we mean a vertex stabilizer in a very small simplicial F_N-tree. We associate to $g \in F_N$, respectively $A \leq F_N$, the smallest free factor containing it, which we denote by $Fill(g)$, respectively $Fill(A)$; g (A) is *simple* if $Fill(g)$ $(Fill(A))$ is a proper subgroup of F_N, hence we get a map $Fill : \{non-trivial\ simple\ elements\ (subgroups)\} \to \mathcal{F}$.

Lemma 5.1. *There is a constant C such that for every very small simplicial F_N-tree T, the set of simple elements fixing a point of T map under Fill to a set of diameter at most C in $\mathcal{F}$.*

We assume that the reader is familiar with Whitehead's algorithm [22]. We use the following notation: for a basis $\mathcal{B}$ for F_N and a cyclically reduced word w over $\mathcal{B}^{\pm}$, let $Wh_{\mathcal{B}}(w)$, or just $Wh(w)$ when $\mathcal{B}$ is understood, stand for the Whitehead graph of w with respect to $\mathcal{B}$; for a subset $\mathcal{B}_0 \subseteq \mathcal{B}$, let $Wh_{\mathcal{B}_0}(w)$ stand for the subgraph of $Wh(w)$ induced by the vertices $\mathcal{B}_0^{\pm}$. Also, let $|w|_{\mathcal{B}}$, or just $|w|$, denote word length of w, and let $|w|_{\mathcal{B}_0}$ denote the total number of occurrences of letters in $\mathcal{B}_0^{\pm}$ in w.

Proof. Let $T \in \partial \mathrm{cv}_N$ be simplicial. If T has an edge e with trivial stabilizer, then collapsing every edge not in the orbit of e gives a tree T' corresponding to a free splitting of F_N, and every simple elliptic element of T is elliptic in T'. The image under *Fill* of the simple elliptic elements of T' has diameter at most 2 in $\mathcal{F}$.

So, assume that T has no edge with trivial stabilizer; collapse edges outside of a fixed orbit of edges and replace T with the resulting 1-edge splitting. This increases the diameter of the image of *Fill* by at most 2. We have two cases to consider, corresponding to whether T/F_N is a segment or a loop.

First suppose that T/F_N is a segment, so T corresponds to a splitting $A *_w B$. By Lemma 4.1 of [1], we have that $A = \langle a_1, \ldots, a_k, w\rangle$ and $B = \langle b_1, \ldots, b_l\rangle$, where $a_1, \ldots, a_k, b_1, \ldots, b_l$ is a basis for F_N; put $\mathcal{A} = \{a_1, \ldots, a_k\}$ and $\mathcal{B} = \{b_1, \ldots, b_l\}$. The $Fill$-image of the set of simple elements contained in A or B has diameter at most 4 in $\mathcal{F}$. Let $g \in A$ be primitive; we assume that $g \notin \langle a_1, \ldots, a_k\rangle$. If $Fill(w)$ is a proper factor of B, then, after choosing a new basis for B, we have that $w \leq \langle b_2, \ldots, b_l\rangle$, so as $\langle a_1, \ldots, a_k\rangle \leq \langle a_1, \ldots, a_k, b_2, \ldots, b_l\rangle \geq \langle g\rangle$, we have that g is of distance at most 2 from A.

So, suppose that $Fill(w) = B$, and apply an automorphism of B to get that $|w|_{\mathcal{B}}$ is minimal; this, along with the condition that $Fill(w) = B$ means that $Wh_{\mathcal{B}}(w)$ is connected with no cut vertex. Let $g \in A$ be primitive and abuse notation, letting g also stand for a cyclically reduced word representing the conjugacy class of g; we assume that w occurs in the (cyclic) word g, else we are done by above. Note that if g does not contain an occurrence of $a_1^{\pm}$, then $B \leq \langle a_2, \ldots, a_k, b_1, \ldots, b_l\rangle \geq \langle g\rangle$ witnesses that g is of distance at most 2 from B. Hence, we assume that g contains an occurrence of each a_i or its inverse.

Apply an automorphism of F_N preserving $\langle a_1, \ldots, a_k\rangle$ up to conjugacy to minimize $|g|_{\mathcal{A}}$. This ensures that $Wh_{\mathcal{A}}(g)$ either is connected with no cut vertex or contains no edge. Let's see that the former case is impossible. If $w = xw_1y$, then $x \neq y^{-1}$, so there are exactly two $\mathcal{B}$-vertices, namely x and y^{-1} that are connected to an $\mathcal{A}$-vertex; hence, no $\mathcal{B}$ vertex can be a cut point for $Wh(g)$. But g is primitive, so $Wh(g)$ must have a cut point, so there is an $\mathcal{A}$-vertex, say c, which is the unique $\mathcal{A}$-vertex connected to a $\mathcal{B}$-vertex. This means that every occurrence of w in g is preceded by c^{-1} and followed by c. So, conjugating B by c reduces $|g|_{\mathcal{A}}$; repeating this argument eventually gives a contradiction, and we conclude that $Wh_{\mathcal{A}}(g)$ cannot be connected and without cut points.

So, consider the latter case, that $Wh_{\mathcal{A}}(g)$ contains no edge. Again, write $w = xw_1y$. Since g contains an occurrence of each a_i, or its inverse, every $\mathcal{A}$-vertex of $Wh(g)$ is connected either to x or y^{-1}; hence $Wh(g)$ is connected. As g is primitive, $Wh(g)$ contains a cut point. But $Wh_{\mathcal{A}}(g)$ contains no edge, so no $\mathcal{A}$-vertex could be a cut point. Hence, either x or y^{-1} is a cut vertex, and since $Wh_{\mathcal{B}}(g)$ is connected without cut vertices, there must be $c \in \mathcal{A}^{\pm}$ such that there is exactly one edge of $Wh(g)$ incident on c. This means that every occurrence of c^{-1} in the (unoriented, cyclic) word g is followed by an occurrence of w; hence the automorphism φ sending c^{-1} to $c^{-1}w^{-1}$ reduces the number of occurrences of w in g. As φ fixes B, we are done by induction.

To conclude, we need to handle the case where T/F_N is a loop, *i.e.* T

corresponds to an HNN-extension. By Lemma 4.1 of [1], T corresponds to an HNN-extension of the form $\langle a_1, \ldots, a_{N-1}, w^{a_N}\rangle *_w$, where $\{a_1, \ldots a_N\}$ is a basis for F_N. Set $\mathcal{A} = \{a_1, \ldots, a_{N-1}\}$, and let $g \in A$ be primitive. By arguing as when T/F_N is a segment, we reduce to the case where $Fill(w) = A$ and $|w|_{\mathcal{A}}$ is minimal. If w^{a_N} occurs in g, then we argue exactly as in the case above, where $Wh_{\mathcal{A}}(g)$ is connected without cut points to arrive at a contradiction. Hence, g must be contained in A, and we are done. ☐

To extend Lemma 5.1 to all trees in $\partial \mathrm{cv}_N$, we use the following result. Recall that for $T \in \partial \mathrm{cv}_N$, $\mathcal{R}(T)$ denotes the collection of all factors reducing T.

Proposition 5.2. *[24, Theorem 1.3] Let $T \in \partial cv_N$, and assume that T is not arational. There is a simplicial tree T_0 such that for any $F \in \mathcal{R}(T)$, some element of F fixes a point in T_0.*

It follows that the diameter of $\mathcal{R}(T)$ in $\mathcal{F}$ is at most two more than the diameter of the $Fill$-image of the set of simple elliptic elements in T_0, hence we get:

Corollary 5.3. *Let $T \in \partial cv_N$, and assume that T is not arational. The set $\mathcal{R}(T)$ has uniformly bounded diameter in $\mathcal{F}$.*

6 Sequences of Geodesics

We now begin our study of the projection $\pi : CV_N \to \mathcal{F}$.

6.1 Limits of Sequences of Geodesics

Fix a basis $\mathcal{B}$ for F_N; for $g \in F_N$, let $|g|$ denote the word length of g in $\mathcal{B}$. We work in unprojectivised Outer space cv_N, and the following is essential for the remainder of the paper:

Remark 6.1. All (projectivized) currents come from ($\mathbb{P}M_N$) M_N.

For a tree $T \in \overline{\mathrm{cv}_N}$ and for $g \in F_N$, we use $\langle T, g\rangle$ to mean $\langle T, \eta_g\rangle$, which is the translation length of g in T by Proposition 2.1.

Suppose we have a sequence of folding lines $S_n \to T_n$ between free simplicial trees, without parametrizing. We assume that T_n converges to T and that S_n/λ_n converges to S for some $\lambda_n \in \mathbb{R}$.

Lemma 6.2. *In this situation,* $\inf \lambda_n > 0$.

Proof. Fix some $g \in F_N$. Then $\langle S_n, g\rangle \geq \langle T_n, g\rangle$ i.e. $\lambda_n \langle S_n/\lambda_n, g\rangle \geq \langle T_n, g\rangle$. Passing to the limit and assuming $\lambda_n \to 0$ gives $\langle T, g\rangle = 0$, which is impossible, since g was arbitrary. $\square$

Use T^* to denote the set of (projectivized) currents ν with $\langle T, \nu\rangle = 0$, so $T^* = \emptyset$ if and only if T is free and simplicial.

Lemma 6.3. *If λ_n is bounded above then $T^* \supseteq S^*$.*

Proof. We may take $\lambda_n = 1$. As above, for any $g \in F_N$ we have $\langle S, g\rangle \geq \langle T, g\rangle$. Let ν be some current, and let $g_n \in F_N$ be such that $g_n/|g_n| \to \nu$. Passing to the limit and using continuity of $\langle \cdot, \cdot\rangle$ gives that $\langle S, \nu\rangle = 0$ implies that $\langle T, \nu\rangle = 0$. $\square$

The same line of reasoning shows:

Lemma 6.4. *Let $S, T \in \overline{cv}_N$. If there is a Lipschitz map $S \to T$, then $S^* \subseteq T^*$.*

To the sequence of folding lines $\{S_n \to T_n\}$ we associate a closed subset of projectivized measured currents $\mathcal{C}(S_n \to T_n)$ defined as the set of (projective classes of) those ν that can be represented as $\lim \gamma_n/|\gamma_n|$ where γ_n is a legal loop in S_n. Without further comment, we always allow passing to a subsequence of $S_n \to T_n$.

Lemma 6.5. *Let $S_n \to T_n$ be folding lines such that T_n converges to T without scaling and such that S_n/λ_n converges to S. If $\lambda_n \to \infty$, then $\mathcal{C}(S_n \to T_n) \subseteq S^*$.*

Proof. Since γ_n is legal we have $\langle S_n, \gamma_n\rangle = \langle T_n, \gamma_n\rangle$ i.e.

$$\langle S_n/\lambda_n, \gamma_n/|\gamma_n|\rangle = \frac{1}{\lambda_n}\langle T_n, \gamma_n/|\gamma_n|\rangle$$

On the other hand, continuity of $\langle \cdot, \cdot\rangle$ gives $\langle T_n, \gamma_n/|\gamma_n|\rangle \to \langle T, \nu\rangle < \infty$, so $\langle S, \nu\rangle = 0$. $\square$

The folding lines $S_n \to T_n$ and $S \to T$ are weak analogues of Teichmüller geodesics; one thinks of T^* as a vertical foliation, and of $\mathcal{C}(S_n \to T_n)$ as a horizontal foliation. Standard geodesics are not quite as well-behaved as folding lines, so we establish Lemma 6.6 below to relate the "left" endpoints of a sequence of standard geodesics to the "left" endpoints of the corresponding folding lines.

Lemma 6.6. *Let $S_n \to U_n \to T_n$ be a sequence of standard geodesics,* i.e. *$U_n = (S_n)_{T_n}$, where S_n/λ_n converges to S, U_n/ρ_n converges to U, and T_n converges to T. There is a tree V and equivariant, Lipschitz maps with convex point pre-images $f_S : V \to S$ and $f_U : V \to U$.*

Proof. Let γ_n be a geodesic path (in a simplex) $S_n/\lambda_n \to U_n/\rho_n$, and let V'_n denote the midpoint of γ_n. Then V'_n admits obvious Lipschitz maps to S_n/λ_n and U_n/ρ_n. Replace the metric on V'_n with the sum of the pull-backs of the metrics from S_n/λ_n and U_n/ρ_n to get a tree V_n, *i.e.* $V_n = 2V'_n$. Clearly V_n converges to a tree V without rescaling.

On the other hand, as $l_{V_n} - l_{S_n/\lambda_n}$ and $l_{V_n} - l_{U_n/\rho_n}$ are length functions for free simplicial trees, $l_V - l_S$ and $l_V - l_U$ are length functions for very small trees; it follows that there are 1-Lipschitz maps $V \to U$ and $V \to S$. □

Lemma 6.7. *Let $S_n \to U_n \to T_n$ be a sequence of standard geodesics such that S_n/λ_n converges to S, U_n/ρ_n converges to U, and T_n converges to T. If $\lambda_n \to \infty$ and λ_n/ρ_n is bounded, then $\mathcal{C}(U_n \to T_n) \subseteq S^*$.*

Proof. Let V as in the conclusion of Lemma 6.6. If λ_n/ρ_n is bounded, then there is a Lipschitz map $S \to V$, hence a Lipschitz map $S \to U$, and the conclusion follows from Lemma 6.5 and Lemma 6.4. □

The following result gives the needed control over limits of sequences of folding lines.

Theorem 6.8. *Let $S_n \to U_n \to T_n$ be a folding path with $S_n/\lambda_n \to S$, $U_n/\rho_n \to U$, $T_n \to T$. Then*

(i) If ρ_n is bounded, $T^* \supseteq U^*$; in particular, T is not free simplicial if U is not.

(ii) If ρ_n is not bounded, then $S^* \cap U^* \neq \emptyset$.

Note that by Lemma 6.2 the sequences $\{\lambda_n\}, \{\rho_n\}, \{\lambda_n/\rho_n\}$ are bounded away from 0, so the two possibilities listed in (i) and (ii) are the only ones.

Proof. The conclusion in (i) follows from Lemma 6.3. In case (ii) we may take $\rho_n \to \infty$ and therefore $\lambda_n \to \infty$. Then observe that $\mathcal{C}(S_n \to T_n) \subseteq \mathcal{C}(U_n \to T_n)$ and by Lemma 6.5 $\mathcal{C}(S_n \to T_n) \subseteq S^*$ and $\mathcal{C}(U_n \to T_n) \subseteq U^*$. Since $\mathcal{C}(S_n \to T_n) \neq \emptyset$, we have $S^* \cap U^* \neq \emptyset$. □

Lemma 6.9. *Suppose that S_n/λ_n converges to S, that T_n converges to T, and let $S_n \to T_n$ be a folding path. If $S^* = T^*$, then any tree U representing a point in the accumulation set of $S_n \to T_n$ in $\overline{CV}_N$ satisfies $\emptyset \neq U^* \subseteq T^*$.*

Note that since we require elements of T^* to come from the minset of currents, Theorem 4.4 gives that $S^* \neq T^*$ implies that $S^* \cap T^* = \emptyset$.

Proof. Let $S_n \to U_n \to T_n$ be subdivisions of the folding paths such that U_n/ρ converges to U, and suppose $S^* = T^*$. Note first that if λ_n are bounded, then there are Lipschitz maps $S \to U$ and $U \to T$, so by Lemma 6.4, we have $S^* \subseteq U^* \subseteq T^*$.

Assume that $\lambda_n \to \infty$. By above, we can find $\gamma_n \in F$ such that γ_n is legal in S_n and such that $\gamma_n/|\gamma_n| \to \mu \in S^* = T^*$. By above, if $\rho_n \to \infty$, then U^* is non-empty, so suppose that ρ_n are bounded; we assume $\rho_n = 1$. In this case we see that $\langle U_n, \gamma_n/||\gamma_n||\rangle = \langle T_n, \gamma_n/||\gamma_n||\rangle$, hence $\langle U, \mu\rangle = 0$, *i.e.* $U^* \neq \emptyset$. Since the same argument goes through for subsequences of $S_n \to T_n$, the claim follows. $\square$

Lemma 6.10. *Suppose $S_n/\lambda_n \to S$, $T_n \to T$, and let $S_n \to T_n$ be a folding path. If S and T are arational with $S^* \neq T^*$, then $S_n \to T_n$ accumulates on some subset of cv_N.*

Proof. We find a sequence U_n such that $S_n \to U_n \to T_n$, where $U_n/\rho_n \to U$ and $U^* = \emptyset$. Note that λ_n cannot be bounded, as $S^* \cap T^*$ is empty. By Theorem 6.8, it is necessary to ensure that ρ_n are bounded. Choose parametrizations $l_n : [0, d_n] \to cv_N$ of $S_n \to T_n$ that project to geodesics in CV_N, and fix $M > 1$. Choose a basis $B = \{b_1, \ldots, b_N\}$ for F, and define $t_n := \inf\{t \in [0, d_n] | \max\{l_{l_n(t)}(b_i)\} \leq M \max\{l_{T_n}(g_i)\}\}$ and put $U_n := l_n(t_n)$. Then, after possibly passing to a subsequence, U_n converges in cv_N to a non-trivial action U (non-triviality of U follows from non-triviality of T). By above $U^* \subseteq T^*$.

Now if for all M and all U_n as above, we have that $U^* = T^*$, then we get that the accumulation set X of the paths $S_n \to T_n$ in compactified Outer space consists entirely of trees V such that either $V^* = S^*$ or $V^* = T^*$. On the other hand, X is evidently connected, whereas $\{V \in \overline{CV}_N | V^* = T^*\} \cup \{V \in \overline{CV}_N | V^* = S^*\}$ is not connected, so X must contain a point of CV_N. $\square$

6.2 Reducing Factors are Visible

We now work in Outer space, so all trees have co-volume 1; recall that here folding line means the image of a folding line as constructed and used above.

Lemma 6.11. *Let $T \in \overline{cv}_N$, and let Σ be a fixed partial simplex in CV_N whose closure does not contain T. There exists a point T_0 in Σ such that*

any optimal map $f : T_0 \to T$ induces a train track structure on T_0 such that there is a legal path that crosses every edge of T_0/F_N.

Proof. Let $T_0 \in \Sigma$ be such that $Lip(f : T_0 \to T)$ is minimized over Σ, such a point exists, since the function $d : \Sigma \to \mathbb{R} : Y \mapsto d(Y, T)$ is clearly proper. Let $G = T_0/F_N$. If there is no legal path crossing every edge of G, then there is a subgraph B of G such that any legal path entering B cannot exit; call a minimal such subgraph a *box*. Let B be a box; say that an edge e of B is *flippable* if there is a legal path in B crossing e in both orientations, and say that e is *unflippable* otherwise.

Let E be the set of edges outside of B that are incident on B and which make a legal turn with an edge of B; necessarily $E \neq \emptyset$, and all edges of B that make a legal turn with $e \in E$ are unflippable. By the definition of a box, specifying an orientation on one edge incident on $e \in E$ and making a legal turn with e by declaring e as incoming specifies an orientation on B.

We now perform a backwards flow relative to this orientation of B. For any $x \in B$, flow x along B in the direction opposite to the orientation of B for time ϵ; this gives a modified map $f_\epsilon : G \to T$, which is well-defined even for vertices of B and for points x incident on $e \in E$. Consider an edge $e \in E$, and note that the endpoint x of e incident on B is moved to the same location (on an edge of B) as it would have been moved to after folding for time ϵ, hence the Lipschitz constant of f_ϵ on this edge is less than that of f. On the other hand, the Lipschitz constant has not increased on the remainder of G, which is a contradiction, since we can now pull-back the measure from T via f_ϵ to G to get $G' \in \Sigma$ such that $d(\tilde{G}', T) < d(T_0, T)$. $\square$

Lemma 6.12. *Let $\Delta_0, \Gamma_i \in CV_N$ and assume Γ_i converges to $T \in \partial CV_N$. Let $\gamma_i = \alpha_i\beta_i$ be a standard geodesic path from Δ_0 to Γ_i: β_i a folding path, and α_i a path in a simplex. Denote by Δ_i the common endpoint of α_i and β_i. Then one of the following holds, after a subsequence.*

(i) Δ_i converges to $\Delta \in CV_N$ and certain initial segments of β_i converge uniformly on compact sets to a folding path (ray) γ from Δ that converges to $S \in \partial CV_N$ with $S^* \subseteq T^*$, or

(ii) Δ_i converges to a tree $S \in \partial cv_N$, and $S^* \subset T^*$.

Proof. Let $f_i : \Delta_0 \to \Gamma_i$ be optimal maps, so Δ_i is obtained from Δ_0 by pulling-back the measure on Γ_i via f_i. If a subsequence of Δ_i project to a sequence contained in a compact subset of CV_N, then we are in case (i). Else, the injectivity radii of Δ_i go to zero, so that after a subsequence $\Delta_i \to S$. We now show that $S^* \subseteq T^*$. As Δ_i degenerate to S, there are subgraphs

$G_i \subseteq \Delta_i$ such that $\pi_1(G_i) \neq \{1\}$ and for $i \neq j$, $\pi_1(G_i)$ can be identified with $\pi_1(G_j)$, and such that $vol(G_i) \to 0$.

First note that if the Lipschitz distances from Δ_0 to Γ_i are bounded, then we get a Lipschitz map $S \to T$, and we are done by Lemma 6.4. So, suppose this is not the case. Pass to a subsequence so that all train track structures on Δ_i agree, and note that the hypotheses of Lemma 6.11 are satisfied; hence there is an element $g \in F_N$ whose representatives in Δ_i are legal and cross every edge of Δ_i. Let s be a loop contained in G_i, then $length_{\Gamma_i}(s)/length_{\Gamma_i}(g) \leq length_{\Delta_i}(s)/length_{\Delta_i}(g) \to 0$; hence $S^* \subseteq T^*$.

Now suppose that we are in case (i), *i.e.* after a subsequence, Δ_i converge to $\Delta \in CV_N$. If we show the convergence statement, then the claim $S^* \subseteq T^*$ will follow from Theorem 6.8. It follows from the Arzéla-Ascoli Theorem that any subsequence of β_i subsequentially converges uniformly on compact sets to a ray r_t in CV_N; we show that there is a subsequence whose subsequential limit r_t is a folding path. The point here is that being a folding path is a local condition.

For $t \in [0, \infty)$, we have that $\beta_i(t)$ converge to r_t. Pass to a subsequence so that $\beta_i(t)$ all lie in the same open simplex in CV_N and so that the train track structures on β_i all agree. If r_t is contained in the same open simplex, then r_t has a well-defined train track structure, and we see that $r|_{[t,t+\epsilon)}$ is a folding path as desired. If r_t is contained in a face of the open simplex containing $\beta_i(t)$, then for some forest of $\beta_i(t)$ is collapsed in the limit; for $i >> 0$ the maximal diameter δ of a component of this forest is small compared to ϵ, where ϵ is chosen to be half of the distance from r_t to a face of the open simplex containing it. Now just note that we get a well-defined train track structure on r_t: a turn is illegal in r_t if it is approximate by a pair of edges exiting the forest being collapsed in $\beta_i(t)$ that are folded at time $t + 2\delta$.

Note that the sequence of times t_k for which we need to pass to a subsequence to ensure that $\beta_i(t_k)$ are contained in a fixed open simplex must satisfy $t_k \to \infty$, else we are in case (ii). Further, the only way the train track structures on $\beta_i(t)$ could not agree for $i >> 0$ is if $\beta_i(t)$ converge to a face of the open simplex containing them. It easily follows that there is a subsequential limit r of β_i that is a folding line, as desired. $\square$

From Lemma 6.12, one has that if T is represented by arational trees, and if Δ and Γ_i are as in the statement, then we always are in case (i).

Lemma 6.13. *Suppose G_t, $t \in [0, \infty)$, is a folding path converging to $T \in \partial cv_N$, and assume that T is not arational. Then there is a factor $F' \leq F_N$ such that $F' \in \mathcal{R}(T)$, and such that F' has uniformly bounded volume along G_t for t large.*

In particular, the projection of G_t to $\mathcal{F}$ is bounded and is uniformly bounded for large t.

Proof. Let A be any factor reducing T. It follows from [3, Lemma 4.1] that $A|G_t$ cannot contain a legal segment of length > 2 inside a topological edge; otherwise the volume of $A|G$ would grow exponentially, and $A|T$ would not have dense orbits.

So now there are two possibilities: (1) If the number of illegal turns in $A|G_t$ is uniformly bounded for large t, then $A|G_t$ has uniformly bounded volume and $d_{\mathcal{F}}(A, G_t)$ is uniformly bounded by [3, Corollary 3.5]. In particular, we are done in this case. (2) For large t, $A|G_t$ has a topological edge with M illegal turns, and these turns do not become legal under further folding.

Assume that we are in case (2). Choose t_0 such that the number of illegal turns in $A|G_t$ has stabilized for $t \geq t_0$. The total number of illegal turns in any train track is uniformly bounded, so for M large some illegal turn appears twice; this gives a loop $g'|G_t$ of uniformly bounded length for t large. Each such loop in G_t lifts uniquely to a loop $g'|G_{t_0}$, and our assumptions ensure that $g'|G_{t_0}$ are all uniformly bounded. Since there are finitely many short loops in G_{t_0}, some such loop $g|G_{t_0}$ is uniformly bounded in G_t for $t \to \infty$.

For $t_n \to \infty$, we have scaling constants λ_n such that $\lambda_n \tilde{G}_{t_n}$ converges to $T \in \partial \mathrm{cv}_N$; if $\lambda_n \to 0$, then g is elliptic in T. If λ_n do not converge to 0, then T contains a simplicial edge with trivial stabilizer. Indeed, in this case there are K-Lipschitz maps $G_{t_0} \to G_{t_n}$ for all n, and this can happen only if there is a non-degenerate arc I contained in an edge of G_{t_1} that is never folded for $t > t_1$. In this case, the complement of I is a factor that is uniformly bounded for $t >> 0$, and this factor certainly contains a subgroup reducing T, and we are done. Hence, we suppose that g is elliptic in T; if g is simple, we are done. We will construct from g a simple element which is uniformly bounded along G_t; to do this we will need the following:

Claim 6.14. If G_t is a folding line that converges to $T \in \partial \mathrm{cv}_N$ and if there is a non-degenerate arc $I = [x, y] \subseteq T$ with non-trivial stabilizer, then either $Stab(I) = Stab(x)$ or $Stab(I) = Stab(y)$.

Proof of Claim 6.14. Choose $a \in Stab(x)$, $b \in Stab(y)$, and let c generate $Stab(I)$; so ab and bc are also elliptic in T. We will show that ac is also elliptic, which proves the claim.

Note that if $g \in F_N$ is elliptic in T, then the length of $g|G_t$ is necessarily bounded; indeed, the number of illegal turns in $g|G_0$ is an upper bound for

the number of illegal turns in $g|G_t$, so if $g|G_t$ is unbounded, then $g|G_t$ must contain a long legal segment. Choose a basepoint $b \in G_0$; the images b_t of b in G_t give basepoints in G_t. We think of all elements of F_N as loops based at b_t. Choose graphs $H^t_a, H^t_b, H^t_c, H^t_{ab}, H^t_{bc}$ with immersions into G_t representing a, b, c, ab, bc, respectively; each H^t_i looks like a balloon, *i.e.* a circle, possibly with a (long) segment, called a *string*, attached to it. After contracting the strings to a point, we get graphs of bounded size for all t.

If all strings are short, then bc is clearly represented by immersions of bounded size for all t, implying that bc is elliptic in T. If the strings for H^t_a are not short, then since $ab|G_t$ is bounded, the string for H^t_a contains all but a bounded amount of the string for H^t_b for all t; similarly, the string for H^t_b contains all but a bounded amount of the string for H^t_c. Hence the string for H^t_a contains all but a bounded amount of the string for H^t_c, and it follows that ac is bounded in G_t as well and is elliptic in T. □

The proof of the following claim uses the Rips Theory; see [2] for background.

Claim 6.15. If for any non-degenerate arc $I = [x, y] \subseteq T$, we have that either $Stab(I) = Stab(x)$ or $Stab(I) = Stab(y)$, then there is at most one maximal cyclic subgroup of F_N that is not simple and is elliptic in T.

Proof of Claim 6.15. If there is an arc $I \subseteq T$ with non-trivial stabilizer, then we can collapse the parts of T with dense orbits to get a simplicial tree T'; see [21] or [24]. Since T' is minimal, using the classification of very small simplicial trees (see *e.g.* [24]), we see that T' must contain a non-degenerate arc J with trivial stabilizer; hence the same holds for T. It follows that every elliptic element is simple in T; indeed, we can collapse the components of $T \smallsetminus F_N J$ to points to get a simplicial tree with trivial arc stabilizers, and every subgroup fixing a point in T fixes a point in this tree as well.

So, assume that T has trivial arc stabilizers. If T does not have dense orbits, then we are done by the same argument as in the previous paragraph. Assume that T has dense orbits. Let $Y \to T$ be a resolution of T by a geometric tree Y such that every subgroup fixing a point in T fixes a point in Y. If T is not geometric, then Y contains a simplicial edge, whose stabilizer must be trivial, and we are done by above. If Y contains a thin component, then running the Rips machine on a band complex to which Y is dual eventually produces a band that is disjoint from any loop contained in a leaf, hence all point stabilizers are simple. It follows that for Y to contain a non-simple elliptic subgroup Y must be dual a measured foliation on a

surface. The claim is then a consequence of well-known facts about surface foliations. □

Back to the proof of Lemma 6.13. Let g be the uniformly bounded element constructed above, and suppose that g is not simple; by the above claims, this means that g represents the boundary of a surface, which carries a measured foliation to which T is dual. If the volume of $A|G_t$ is not uniformly bounded and if there is no simple class that is uniformly bounded in $G|t$, then the pigeon hole principle gives that $A|G_t$ consists of long segments that are labeled by high powers of g attached to a forest of uniformly bounded volume, and there is a uniformly bounded number of such segments.

Since we constructed g by using repeated illegal turns in $A|G_t$ cutting out of $A|G_t$ the segments labeled by maximal powers of g gives a graph $B|G_t$ of uniformly bounded size. We want to see that $B \leq F_N$ is simple. Note that B is finitely generated. Since B has uniformly bounded volume along G_t, B reduces T, and the only way that B is not simple is if B contains a loop representing a power of the boundary component of the surface. This is impossible, since, if B contained such a loop, then by the construction of B, A would contain such a loop. Let $C = Fill(B)$, then the immersion into G_t representing B factors through the immersion representing C. Since $C = Fill(B)$ the immersion representing $B \leq C$ crosses every edge of the immersion representing $C \leq F_N$ in G_t. Hence, if C is contains a long edge, then so does B, which is impossible. It follows that $C|G_t$ has uniformly bounded volume, and C reduces T.

□

7 The Boundary of the Complex of Free Factors

Let $\partial\mathscr{F}$ denote the boundary of the complex of free factors, and let $\mathcal{AT} \subseteq \partial CV_N$ denote the set of (classes of) arational trees.

Define an equivalence relation $\sim$ on $\mathcal{AT}$, where $T \sim S$ if and only if $L(T) = L(S)$; we note that $\sim$ is precisely the relation of "forgetting the measure" for elements of $\mathcal{AT}$; see [8]. Give $\mathcal{AT}$ the subspace topology, and consider the quotient space $p : \mathcal{AT} \to \mathcal{AT}/\sim$.

Lemma 7.1. *The quotient map $p : \mathcal{AT} \to \mathcal{AT}/\sim$ is closed, and point pre-images are compact.*

Proof. Let $K \subseteq \mathcal{AT}$ be closed; we show that $C = p^{-1}(p(K))$ is closed. Let $\{T_n\}$ be a convergent sequence in C, say T_n converges to $T \in \mathcal{AT}$;

let $Y_n \in K$ such that $p(Y_n) = p(T_n)$. This means that $L(Y_n) = L(T_n)$, or, equivalently, that $Y_n^* = T_n^*$. Now, let $\eta_n \in Y_n^* = T_n^*$. After passing to a further subsequence we may assume that $Y_n \to Y \in \partial \mathrm{cv}_N$ and that $\eta_n \to \eta \in M_N$. By Proposition 2.1, we have $\langle Y, \eta \rangle = 0 = \langle T, \eta \rangle$; since $T \in \mathcal{AT}$, and since Proposition 2.2 gives $Supp(\eta) \subseteq L(T), L(Y)$, Theorem 4.4 gives that Y is arational and that $L(T) = L(Y)$. It follows that $Y \in K$ and $p(T) = p(Y)$, so $T \in C$.

The statement that equivalence classes are compact can be proved similarly using the compactness of $\partial \mathrm{cv}_N$. If T_i converge to T in $\partial \mathrm{cv}_N$ and if $T_i \in \mathcal{AT}$ are all equivalent, then choose some $\nu \in T_i^*$. By Proposition 2.1 we have $\nu \in T^*$ and then $T \in \mathcal{AT}$ is equivalent to all T_i by Theorem 4.4. $\square$

The following result justifies our use of sequential arguments.

Corollary 7.2. *The quotient space $\mathcal{AT}/\sim$ is separable and metrizable.*

Proof. First, $\mathcal{AT}/\sim$ is normal; using the fact that $\partial \mathrm{cv}_N$ is metrizable and normal, this is a not completely obvious exercise in point set topology, which is done in [20, Proposition 7.2]). Since $\mathcal{AT}$ is separable, so is $\mathcal{AT}/\sim$. Next, observe that $\mathcal{AT}/\sim$ is second countable: if $\{U_1, U_2, \cdots\}$ is a basis of open sets in $\mathcal{AT}$, the collection of sets

$$(\mathcal{AT}/\sim) - \partial\pi(\mathcal{AT} - (U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_k}))$$

is a countable basis of open sets in $\mathcal{AT}/\sim$. Now Urysohn's Lemma implies that $\mathcal{AT}/\sim$ can be embedded in $[0,1]^\infty$ and is therefore metrizable. $\square$

We can now give a description of $\partial\mathcal{F}$.

Proposition 7.3. *There is a continuous map $\partial\pi : \mathcal{AT} \to \partial\mathcal{F}$, such that if $T_i \in CV_N$ converge to $T \in \mathcal{AT}$, then $\pi(T_i)$ converges to $\partial\pi(T)$.*

Proof. Let $T_i \in CV_N$ converge to T; we need to see that $\pi(T_i)$ converges to a point of $\partial\mathcal{F}$. Toward contradiction, suppose this is not the case, then we get subsequences X_n and Y_n such that $(\pi(X_n)|\pi(Y_n))$ is uniformly bounded. Consider a standard geodesic $X_n \to Y_n$; Proposition 2.5 gives that these geodesics are mapped by π to uniform quasi-geodesics in $\mathcal{F}$. Hence we find Z_n on $X_n \to Y_n$ with $\pi(Z_n)$ of uniformly bounded distance from any basepoint in $\mathcal{F}$. On the other hand, Lemma 6.9 and Theorem 4.4 give that any limit Z of $\{Z_n\}$ must be arational. Finally, Corollary 4.5 gives a contradiction.

The continuity statement follows similarly, let $T_i \in \mathcal{AT}$ converge to $T \in \mathcal{AT}$, and choose $Y_k^n \in CV_N$ such that Y_k^n converges to T_n and such

that Y^n_n converges to T. By the previous paragraph, $\pi(Y^n_k)$ converges to $\partial\pi(T_n)$, and $\partial\pi(Y^n_n)$ converges to $\partial\pi(T)$. If it is not the case that for n, n', k, k' large we have that $(\pi(Y^n_k)|\pi(Y^{n'}_{k'}))$ is large, then after passing to subsequences, $(\pi(Y^n_k)|\pi(Y^{n'}_{k'})$ is uniformly bounded. Now apply Lemma 6.9 to the standard geodesics $Y^n_k \to Y^{n'}_{k'}$ to get that these geodesics only accumulate on trees $U \in \mathcal{AT}$ such that $U^* = T^*$; finally, apply Corollary 4.5 to get a contradiction. □

Proposition 7.4. *For arational trees S and T, we have $\partial\pi(S) = \partial\pi(T)$ if and only if $L(S) = L(T)$*

Proof. By the same argument as in the proof of Proposition 7.3, we get that $L(S) = L(T)$ implies that $\partial\pi(S) = \partial\pi(T)$. So assume that $L(S) \neq L(T)$, let S_n converge to S and T_n converge to T; consider standard geodesics $S_n \to T_n$. By Lemma 6.10, we have that $S_n \to T_n$ accumulates on some portion of CV_N, hence after passing to a subsequence, we find points on $S_n \to T_n$ projecting to points of $\mathcal{F}$ of uniformly bounded distance from any base point. Hence $(\pi(S_n)|\pi(T_n))$ is uniformly bounded, so $\partial\pi(S) \neq \partial\pi(T)$. □

Proposition 7.5. *The map $\partial\pi$ is surjective. Further, if $\{T_n\}$ converge to a tree T that is not arational, then no subsequence of $\{\pi(T_n)\}$ converges to a point of $\partial\mathcal{F}$.*

Proof. Let $X \in \partial\mathcal{F}$, and let $X_n \in \mathcal{F}$ converge to X. Choose $T_n \in \pi^{-1}(X_n)$, and pass to a subsequence so that $\{T_n\}$ converges to T in $\overline{CV}_N$. We will show that $T \in \mathcal{AT}$, which implies $\partial\pi(T) = X$.

Toward contradiction, suppose that T is not arational. Consider for each n and $m > n$ a standard geodesic $T_n \to T_m$, and let $m \to \infty$. Apply Lemma 6.12, and first assume that case (ii) applies. Then a subgraph of T_n/F_N represents a factor A reducing T, and all T_m project to a uniformly bounded subset of $\mathcal{F}$, as A has uniformly bounded volume in $T_n \to T_m$.

Now, suppose that case (i) of Lemma 6.12 applies, so after a subsequence initial segments of $T_n \to T_m$ converge to a ray r_n that converges to S with $S^* \subseteq T^*$. Note that S^* is not arational, so there is a factor A reducing S^*. Using Lemma 6.13, we see that the projection of r_n to $\mathcal{F}$ is eventually contained in a uniformly bounded neighborhood of $\mathcal{R}(S)$, and therefore of $\mathcal{R}(T)$.

To conclude just note that the projections of $T_n \to T_m$ are uniform quasi-geodesics by Proposition 2.5, so we have for any fixed n infinitely m such that $(X_n|X_m)$ is uniformly bounded, a contradiction. □

Lemma 7.6. *The map $\partial\pi : \mathcal{AT} \to \partial\mathcal{F}$ is closed.*

Proof. Let $C \subseteq \mathcal{AT}$ be closed, and let $K = \partial\pi(C)$. Let $X_n \in K$ converge to $X \in \partial\mathcal{F}$; we want to find $Y \in C$ with $\partial\pi(Y) = X$. Choose $T_n \in (\partial\pi)^{-1}(X_n) \cap C$, and pass to a subsequence to ensure that T_n converge to $T \in \partial\mathrm{cv}_N$.

First, we show that if $T \in \mathcal{AT}$, then $\partial\pi(T) = X$. Let r_n be folding lines originating from a fixed simplex of cv_N with $\lim r_n(t) = T_n$. For t large, using Proposition 7.3 we get that $\pi(r_n(t))$ is close to X_n, so for n very large and t super large, $\pi(r_n(t))$ is close to X. On the other hand, for n very large and t super large, we have that $r_n(t)$ is close to T, so again using Proposition 7.3, we get that $\partial\pi(T) = X$, as desired.

To conclude, we need to show that $T \in \mathcal{AT}$. If not, let r_n be folding lines converging to T_n, and apply Lemmas 6.12 and 6.13 to arrive at a contradiction. $\square$

We get:

Theorem 7.7. *The space $\partial\mathcal{F}$ is homeomorphic to the quotient space $\mathcal{AT}/\sim$.*

Proof. The map $\partial\pi : \mathcal{AT} \to \partial\mathcal{F}$ factors through $p : \mathcal{AT} \to \mathcal{AT}/\sim$ to give a continuous, bijective, closed map $\mathcal{AT}/\sim \to \partial\mathcal{F}$. $\square$

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